The elastica with pre-stress due to natural curvature†

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ABSTRACT

The axial buckling behavior is determined for an elastic beam or rod which has a uniform curvature in its natural state, is straightened by pure bending, and clamped at its ends. Buckling can be either identical to the classical two-dimensional behavior determined by Euler, or it can be three-dimensional involving twist and deflection out of the plane of natural curvature depending on the bending and torsional stiffnesses and the natural curvature. While the classical two-dimensional buckling behavior of Euler’s elastica is stable under applied load, the three-dimensional buckling behavior can be stable or unstable. Theoretical and experimental examples are presented illustrating the full range of possibilities.

1. Introduction

Euler’s (1732) elastica is one of the classic solutions of mechanics. His analysis of a straight elastic rod or beam under axial compression, which is clamped or simply supported at its ends, provides a simple formula for the axial load at which the straight configuration loses stability, and it provides the post-buckling behavior demonstrating that the rod remains stable in the sense that an increasing load is required to amplify the buckle. In this paper a new aspect, natural curvature, is added to Euler’s problem which, unlikely as it might seem, can bring about major changes to the buckling behavior. The study of this basic problem is motivated by recent work on the role of the bending energy associated with natural curvature on the stability and transitions of the multiple states of elastic beam and rod structures (Lachenal et al., 2012; Olsen et al., 2013; Miller et al., 2014; Audoly and Seffen, 2015; Reis, 2015; Xu et al., 2015; Mhatre et al., 2021; Mouthuy et al., 2012; Leanza et al., 2022; Lu et al., 2023a, 2023b).

With reference to Fig. 1, consider an elastic rod, which in the unstressed state has a uniform natural curvature, κn, about the i3-axis, that is then straightened by pure bending and clamped at its ends. The left end is fixed, and the right end is free to move in the direction parallel to i2, along which the straightened column is aligned. Kirchhoff’s (1859) three-dimensional (3D) theory of inextensional elastic rods will be used in the present analysis. Kirchhoff’s model reduces to Euler’s 2D model of the elastica if the rod deflections are confined to the (i1, i2) plane. Moreover, if the deflections are constrained to lie in this plane, it will be shown that the natural curvature has no influence on the buckling behavior. The natural curvature comes into play when the rod is free to deflect out of the (i1, i2) plane. In such cases, stability of the perfectly straight rod is lost in the form of a bifurcation mode that couples out-of-the-plane deflection with torsion. The post-bifurcation analysis reveals that the buckling behavior may be stable or unstable, depending on the specific parameters characterizing the rod and whether load or end-shortening are prescribed. Experimental examples of both types of behavior

† This paper is dedicated to Professor Wei Yang, a major contributor to mechanics and to academic leadership on the occasion of his 70th birthday.
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2. The energy of the system

The rod is assumed to be inextensional with $s$ as the distance measured from the left end. The rod is uniform with a cross-section having principal axes aligned with the $(i_1, i_2)$ axes in the straight pre-buckling state. The centroid of the cross-section in the initial state lies along the $i_2$ axis, as depicted in Fig. 1. A righthanded triad of orthogonal unit vectors, $(e_1, e_2, e_3)$, with $e_2$ tangent to the rod, is embedded in the rod at each point along its centroid line, coincident with $(i_1, i_2, i_3)$ in the initial state. Euler angles, $(\alpha, \beta, \gamma)$, are employed at every point along the rod to describe the orientation of the embedded triad to the fixed Cartesian base vectors, $(i_1, i_2, i_3)$, as defined in Fig. 1. The embedded triad in a deformed state is given by

\[
\begin{align*}
e_1 &= \cos \omega + (1 - \cos \beta) \sin \alpha \sin \gamma i_1 + \sin \alpha \sin \beta i_2 + (1 - \sin \omega) \sin \alpha \cos \gamma i_3 \\
e_2 &= \sin \sin \alpha i_1 + \cos \beta i_2 + \sin \omega i_3 \\
e_3 &= (\sin \omega - (1 - \cos \beta) \cos \gamma) i_1 - \cos \sin \beta i_2 + \cos \omega - (1 - \cos \beta) \cos \gamma i_3
\end{align*}
\]  

(2.1)

where the combination $\omega = \alpha + \gamma$ will appear prominently.

In the version of Kirchhoff rod theory used here, the rod is assumed to be inextensional and to have a linear elastic response in bending and torsion with the two principal bending stiffnesses denoted by $B_1$ and $B_3$, and the torsional stiffness denoted by $B_2$. The curvatures of the rod centerline about $i_1$ and $i_3$ are $\kappa_1$ and $\kappa_3$, while the twist per length about $i_2$ is denoted by $\kappa_2$. These are related to the Euler angles by

\[
\begin{align*}
\kappa_1 &= \frac{\cos(\omega - \gamma) \, d\beta}{ds} + \frac{\sin(\omega - \gamma) \sin \gamma \, d\gamma}{ds} \\
\kappa_2 &= \frac{\sin(\omega - \gamma) \sin \gamma \, d\beta}{ds} + \frac{\sin(\omega - \gamma) \sin \gamma \, d\gamma}{ds} \\
\kappa_3 &= \frac{\sin(\omega - \gamma) \sin \gamma \, d\beta}{ds} + \frac{\sin(\omega - \gamma) \sin \gamma \, d\gamma}{ds}
\end{align*}
\]  

(2.2)

The moment carried by the rod at any point along its length is $B_1 \kappa_1 e_1 + B_2 \kappa_2 e_2 + B_3 (\kappa_3 - \kappa_n) e_3$. The energy of the deformed rod is

\[
\Psi = \frac{1}{2} \int_0^L \left\{B_1 \kappa_1^2 + B_2 \kappa_2^2 + B_3 (\kappa_3 - \kappa_n)^2 - 2P(1 - \cos \beta) \right\} ds
\]  

(2.3)

including the potential energy of the load, $-P\Delta$, where

\[
\Delta = -\int_0^L (1 - \cos \beta) ds
\]  

(2.4)

is the end-shortening, i.e., the displacement of the rod’s right end in the negative $x_2$ direction. In the straight pre-buckling state, the energy is entirely due to the natural curvature, $\Psi = B_3 \kappa_n^2 L/2$.

The formulation above is the standard Kirchhoff rod theory for initially straight rods, as employed, for example, by Champneys & Thompson (1996) and other authors. We now depart somewhat from most earlier approaches by adopting a hybrid Lagrangian formulation as laid out by Leanza, et al. (2023) for straight and curved rods which, in addition to the Euler angles, employs the components of the vector measuring the displacement of the centroid line from its initial state. For the inextensional rod, the components of the displacement vector,

\[
u(s) = \tilde{u}_1 i_1 + \tilde{u}_2 i_2 + \tilde{u}_3 i_3 = u_1 e_1 + u_2 e_2 + u_3 e_3.
\]

are related to the Euler angles by

Fig. 1. A uniform elastic rod of length $L$ with uniform natural curvature, $\kappa_n$, is straightened, clamped at it ends, and subject to a compressive axial load, $P$. The left end is clamped, and the right end is also clamped but free to displace parallel to $i_2$. The principal bending axes of the cross-section align with $(i_1, i_3)$ in the unbuckled state; results for the rectangular cross-section shown will be presented. On the right, the Euler angles $(\beta, \gamma)$ determine the orientation of the tangent to the rod centerline and $\alpha$ contributes to the rotation of the cross-section about the centerline.
with

\[
\cos \beta = \sqrt{1 - (du_1/ds)^2 - (du_3/ds)^2}
\]

The end-shortening (2.4) follows directly from (2.5).

### 2.1. 2D buckling with twist suppressed: Reduction to Euler’s elastica

Consider deformations with \( \omega = 0, \gamma = -\pi/2 \), implying no twist, and, by (2.5), the rod deflection is in the \((i_1, i_2)\) plane with \( du_1/ds = \sin \beta \) and one non-zero strain quantity, \( \kappa_3 = d\beta/ds \). The energy of the rod system (2.3) reduces to

\[
\Psi = \frac{1}{2}Bu_3^2L + \frac{1}{2} \int_0^L \left\{ B_3(d\beta/ds)^2 - 2P(1 - \cos \beta) \right\} ds
\]

(3.1)

(The contribution linear in \( d\beta/ds \) integrates to zero because \( \beta = 0 \) at the ends.) Apart from the constant energy contribution due to the natural curvature, this is precisely the energy functional for a rod clamped at both ends which defines the 2D Euler elastica for buckling.

### 3. Bifurcation analysis for an axially compressed clamped rod with natural curvature

In this section, the functional governing stability of the initial straight state will be derived allowing for all possible admissible deflections and twist, and the bifurcation analysis associated with the onset of buckling will be carried out. In the initial straight state \( \beta = 0 \) and, by (2.1), \( \omega = 0 \), but \( \gamma \) is unknown and denoted by \( \gamma_0(s) \). With \( \gamma = \gamma_0 + \Delta \gamma \), anticipate that \((\beta, \omega, \Delta \gamma)\), which vanish at bifurcation, are of the same order immediately after bifurcation. From (2.2), to order \( O^2 \),

\[
\begin{align*}
\kappa_1 &= (\cos \gamma \omega)' + (\sin \gamma \beta)'(\omega - \Delta \gamma) - \sin \gamma \beta \Delta \gamma' + O^3 \\
\kappa_2 &= \omega' - \frac{1}{2} \beta'^2 + O^3 \\
\kappa_3 &= - (\sin \gamma \beta)' + (\cos \gamma \beta)'(\omega - \Delta \gamma) - \cos \gamma \beta \Delta \gamma' + O^3
\end{align*}
\]

where \((\cdot)' = (\cdot)/ds\). Next, expanding the energy (2.3) to second order in \( (\beta, \omega, \Delta \gamma) \), one obtains contributions homogeneous of degree 0, 1, and 2, \( \Psi = \Psi_0 + \Psi_1 + \Psi_2 \), as

\[
\begin{align*}
\Psi_0 &= \frac{1}{2}Bu_3^2L \\
\Psi_1 &= \int_0^L B_3 \kappa_3 \sin(\gamma_0 \beta) \, ds \\
\Psi_2 &= \frac{1}{2} \int_0^L \left\{ B_1(\cos \gamma \beta)^2 + B_2 \omega^2 + B_3(\sin \gamma \beta)^2 - 2 \kappa_3(\cos \gamma \beta)' (\omega - \Delta \gamma) - \cos \gamma \beta \Delta \gamma' \right\} ds
\end{align*}
\]

(3.2)

First, note that \( \Psi_1 = 0 \) because \( \beta = 0 \) at \( s = 0, L \). Then, noting that \( u_1 = \sin \gamma \beta + O^2 \) and \( u_3 = \cos \gamma \beta + O^2 \), and that \((\cos \gamma \beta)' \Delta \gamma + \cos \gamma \beta \Delta \gamma' = (\cos \gamma \beta) \Delta \gamma' \) integrates to zero, it follows that the quadratic functional governing the energy change from the straight state is

\[
\Psi_2 = \frac{\Psi_2 L}{B_3} = \frac{1}{2} \int_0^L \left\{ b_1 u_3^2 + b_2 \omega^2 + u_1^2 - 2 \kappa_3 u_1' \omega + \Phi(u_1^2 + u_3^2) \right\} dx
\]

(3.3)

with \( x = s/L, \) \( b_1 = B_1/B_3, b_2 = B_2/B_3, \) \( \kappa_3 = B_3L, \) \( \Phi = PL^2/B_3, \) \( u_i/L \rightarrow u_i, \) \( u_3/L \rightarrow u_3, \) \( \omega = 0 \) at \( x = 0 \) and \( u_1, \omega \), \( u_1, u_3, u_3, \omega \), \( 0 = 0 \) at \( x = 1 \)

(3.4)

The classical criterion for the stability of the straight equilibrium state is based on \( \Psi_2 \) being positive for all non-zero admissible combinations of \( (u_1, u_3, \omega) \). Rigid body motions are excluded by (3.4). The displacement component \( u_3 \) does not appear in \( \Psi_2 \); it is auxiliary and can be obtained after the other components are in hand. The quadratic functional \( \Psi_2 \) governs the bifurcation eigenvalue problem. The lowest eigenvalue, \( \Phi_c \), determines the onset of buckling and the associated bifurcation mode. The three ordinary differential equations (ODEs) and natural (dynamic) boundary conditions of the eigenvalue problem which render \( \Psi_2 \) stationary with respect to variations in \( (u_1, u_3, \omega) \), subject to constraints of the end conditions (3.4), are obtained by standard calculus of variations procedures.
\[
\delta F_2 = \int_0^1 \left\{ \left[ b_1 u'' + Pu' - \kappa \omega' \right] \delta u_3 + \left[ u'' + Pu' \right] \delta u_1 + \left[ - b_2 \omega' - \kappa \omega \right] \delta \omega \right\} \, dx \\
+ \left[ b_1 u'' - \kappa \omega' \right] \delta u_3 + \left[ - b_1 u'' + \kappa \omega' - Pu' \right] \delta u_3 + \left[ u'' \right] \delta u_1 + \left[ - u'' - Pu' \right] \delta u_1 + \left[ b_2 \omega' \right] \delta \omega = 0
\] (3.5)

Thus, the end conditions are specified by (3.4) and the ODEs on \(0 \leq x \leq 1\) are
\[
u_1'' + Pu'_1 = 0, \quad b_1 \nu_1'' - \kappa \omega' + Pu'_3 = 0, \quad b_2 \omega' + \kappa \omega = 0
\] (3.6)

The ODE and boundary conditions for \(u_1\) are decoupled from those for \((\nu_1, \omega)\) and can thus be solved independently. The solutions to the two independent eigenvalue problems are easily produced. We will only be interested in the lowest eigenvalues and their associated modes. Throughout this paper, the amplitude of the bifurcation mode will be denoted by \(\xi\). For the first problem:

\[
P_c = 4\pi^2 u_1 = \left( \xi / 2\pi \right) (1 - \cos 2\pi x), \quad \omega = 0
\] (3.7)

For the second problem:

\[
P_c = 4\pi^2 b_1 - \kappa_n^2 / b_2 \quad u_1 = 0, \quad \omega = \left( \xi / 2\pi \right) (1 - \cos 2\pi x), \quad \omega = -\xi (\kappa / 2\pi b_2) (1 - \cos 2\pi x)
\] (3.8)

To order \(\xi\), \(u_2 = 0\) in both problems.

In the first problem, (3.7), the rod buckles in the plane of the natural curvature, i.e., perpendicular to \(i_3\), with no twist or dependence on \(\kappa\). In the second problem the rod buckles perpendicular to the plane of natural curvature accompanied by twist and with a strong dependence on \(\kappa\). Taken together, the two sets of bifurcation results are consistent with the behavior summarized in Fig. 2. If the bending modulus ratio, \(b_1 = B_1 / B_3\), is less than unity, bifurcation is governed by the second problem (3.8) for all values of natural curvature. If, \(b_1 = B_1 / B_3 > 1\), the first problem (3.7) provides the lowest bifurcation eigenvalue if \(\kappa_n^2 < \kappa_n^2\), where \(\kappa_n = 2\pi \sqrt{(b_1 - 1) b_2}\) and the rod buckles in the \((x_1, x_2)\) plane with no twist—this is classical Euler buckling. If, however, \(b_1 > 1\) and \(\kappa_n^2 < \kappa_n^2\), where \(\kappa_n^\text{max} = 2\pi \sqrt{b_1 b_2}\), the rod buckles out-of-plane with twist governed by (3.8). The three domains of behavior are mapped in Fig. 2b. Note that for the second problem \(P_c = 0\) if \(\kappa_n = \pm \kappa_n^\text{max}\). These are the maximum and minimum natural curvatures of a rod such that it can be straightened into a clamped beam without buckling (Leanza et al., 2023). The three domains of behavior are mapped in Fig. 2b. For rods with circular cross-sections, solid or concentric tubular, \(b_1 = 1\) and \(b_2 = 1 / (1 + \nu)\), such that with \(\nu = 1 / 3\), the domain depends only on the natural curvature with limits specified by \(\kappa_n = 0\) and \(\kappa_n^\text{max} = 5.441\).

In the next section, we will discuss the initial post-bifurcation behavior for the entire range of natural curvatures, \(0 \leq \kappa_n^2 \leq \kappa_n^2\), including a discussion of the behavior at the natural curvature limits when buckling occurs under no axial load.

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Fig. 2. a) Bifurcation behavior of the clamped-clamped rod under axial compression as dependent on the modulus ratios, \(b_1 = B_1 / B_3\), \(b_2 = B_2 / B_3\), and dimensionless natural curvature, \(\kappa_n = L \kappa_n\). b) The plot for rectangular cross-sections with \(h/t\) defined in Fig. 1 (and \(\nu = 1 / 3\)) that maps the domains of the dimensionless parameter space, \((L \kappa_n, h/t)\), corresponding to classical 2D Euler buckling and 3D buckling out-of-plane with twist. In the upper left-hand corner of the map, where \(L \kappa_n > 2\pi \sqrt{b_1 b_2}\), the rod is unstable in the straightened state with \(P = 0\).
4. Initial post-bifurcation behavior for out-of-plane displacement with twist

Consider the mode involving both out-of-plane displacement and twist. In the straight state prior to bifurcation, \( \beta = \omega = \gamma = 0 \), where \( \gamma = 0 \) from (2.5) because the bifurcation mode has \( u_1' = 0 \) with \( u_2 \neq 0 \). The energy of the system (2.7) expanded in powers of \( (\beta, \omega, \gamma) \) about the pre-bifurcation state leads to the homogeneous functionals of degrees 0 to 4 listed below. No approximations are made, but terms have been eliminated which integrate to zero owing to \( \beta(0) = \beta(1) = 0 \). Employing the same nondimensionalization used above, one obtains

\[
\Psi = \Psi_0 + \Psi_1 + \Psi_2 + \Psi_3 + \Psi_4 + \ldots
\]

(4.1)

with \( \Psi_0 = \kappa_n^2/2 \), \( \Psi_1 = 0 \),

\[
\Psi_2 = \frac{1}{2} \int_0^1 \left( b_1 \beta' + b_2 \omega' + 2 \kappa_n \beta' \omega' - \beta \Omega \right) dx
\]

\[
\Psi_3 = 0
\]

\[
\Psi_4 = \frac{1}{2} \int_0^1 \left( (b_1 - 1) \left( - (\omega - \gamma)^2 \beta^2 + 2(\omega - \gamma) \beta \gamma + \Omega \right) - b_2 \beta^2 \omega' + \beta^2 \gamma^2 - \kappa_n \left( \frac{1}{3} \beta \gamma + (\omega - \gamma)^2 \beta \omega' \right) + \frac{1}{12} \beta \Omega \right) dx
\]

For the purpose of the initial post-buckling analysis, as an alternative to the use of \( (u_1, u_2, \omega) \) in the earlier section, it is more convenient to use the variables \( (\beta, \omega, \gamma) \) which appear in the expansion above.

As seen in Section 3, the bifurcation eigenvalue problem is governed by \( \Psi_2 \). With variables \( (\beta, \omega, \gamma) \), the ODEs and boundary conditions for the eigenvalue \( \bar{P}_c \) and the associated eigenmode in this formulation are

\[
b_1 \beta' + \beta \Omega - \kappa_n \omega' = 0, b_2 \omega' + \kappa_n \beta' = 0
\]

(4.2)

with \( \beta = 0 \) at \( x = 0, 1 \). For the eigenmode with out-of-plane deflection and twist,

\[
\xi(\Omega^{(1)}, \omega^{(1)}) = \xi(\sin 2\pi x, c_3(1 - \cos 2\pi x)) \quad \text{with} \quad c_3 = -\kappa_n/(2\pi b_2)
\]

(4.3)

and the critical bifurcation eigenvalue is

\[
\bar{P}_c = (2\pi)^2 b_1 - \kappa_n^2/b_2 \quad \text{if} \quad b_1 > 1 \quad \text{and} \quad \kappa_n^2 \leq \kappa_n^{\text{max}}^2
\]

\[
\bar{P}_c = (2\pi)^2 b_1 - \kappa_n^2/b_2 \quad \text{if} \quad b_1 < 1
\]

(4.4)

The mode amplitude is \( \xi \), and associated modal displacements are

\[
\xi(u_1^{(1)}, u_2^{(1)}, \omega^{(1)}) = \xi(0, 0, (1 - \cos 2\pi x)/2\pi)
\]

(4.5)

These are equivalent to the results obtained in the previous section.

The bifurcation functional \( \Psi_2 \) generating the eigenvalue does not involve \( \gamma \). Moreover, \( \Psi_3 = 0 \). The lowest order contribution of \( \gamma, \gamma = \xi^{(1)} \), is obtained by rendering \( \Psi_4 \) stationary with respect to variations in \( \gamma \) resulting in

\[
\beta \gamma^{(1)} + 2 \beta \gamma' + (\kappa_n \omega' - (b_1 - 1) \beta') \gamma = -(b_1 - 1) (\omega' \beta' + b_2 \beta') + \kappa_n \left( \omega \beta' + \frac{1}{2} \beta' \right)
\]

(4.6)

with no restrictions on \( \gamma \) at the ends other than it be bounded. Expressed in terms of the lowest order quantities, this equation becomes

\[
\sin 2\pi x \gamma^{(1)} + 4 \pi \cos 2\pi x \gamma^{(1)} + (2\pi)^2 (b_1 - 1 - b_2 c_3^2) \sin 2\pi x \gamma^{(1)} = (2\pi)^2 c_3 \left[ (b_1 - 1 - b_2 c_3^2) \sin 2\pi x + \right. \left. (b_1 - 1) \frac{1}{2} b_2 (1 + c_3^2) \right] \sin 4\pi x
\]

(4.7)

The homogeneous solution to this equation is obtained by seeking solutions of the form \( \gamma^{(1)} = \sin 2\pi x \gamma(x) \), giving

\[
\gamma^{(1)} = c_1 \frac{\sin px}{\sin 2\pi x} + c_2 \frac{\cos px}{\sin 2\pi x}, \quad \text{with} \quad p = \sqrt{(2\pi)^2 b_1 - \kappa_n^2}/b_2
\]

Because the natural curvature is limited to the range \( \kappa_n^2 < (2\pi)^2 b_1 b_2, p > 0 \). For any non-zero combination of the undetermined coefficients \( c_1 \) and \( c_2 \), the homogeneous solution is unbounded at either \( x = 0 \) or \( x = 1 \), and possibly at \( x = 1/2 \); thus, \( c_1 = c_2 = 0 \). The desired solution is the following particular solution to (4.7):

\[
\gamma^{(1)} = c_3 (1 + \cos 2\pi x) \quad \text{with} \quad r = \frac{2(b_1 - 1) - b_2 (1 + c_3^2)}{4 - b_1 + b_2 c_3^2}
\]

(4.8)

The fact that the cubic terms in \( \Psi \) are zero implies that the initial post-buckling behavior is symmetric with respect to the amplitude of the buckling mode \( \xi \). For sufficiently small \( \xi \),
where $\Delta$ is the end-shortening of the column through which $P$ works. It follows that to lowest order the initial post-buckling relation between the load and the end-shortening is

$$\frac{P}{P_C} = 1 + 4b_2 \Delta/L + \ldots \quad (4.9)$$

We compute $b$ next, but before doing so, it is important to note that if $|\kappa_n| = 2\pi\sqrt{b_1b_2}$, bifurcation occurs with no axial load, i.e., $P_C = 0$, and this limit will be analyzed separately in the next sub-section.

With the lowest order bifurcation results above in hand, the initial post-bifurcation behavior is readily determined using Koiter’s systematic method for expanding in the amplitude $\xi$. The details of the computation of the post-bifurcation coefficient, $b$, are relatively straightforward for systems having $\Psi_3 = 0$. The general result, which one can derive using Koiter’s method for this class of problems, is

$$b = \frac{2\Psi_4(u^{(1)})}{P_1\Psi_3^P(u^{(1)})} \quad (4.11)$$

where $\Psi_3^P$ is the contribution to $\Psi_3$ multiplying $\Psi$, $\xi u^{(1)}$ is the general representation of the bifurcation mode, and, in $\Psi_4(u^{(1)})$, $P$ is equated to $P_C$. The evaluation of the two contributions to $b$ can be carried out analytically:

$$\frac{P_1\Psi_3^P(u^{(1)})}{\Psi_4(u^{(1)})} = \frac{1}{4}(2\pi)^2(b_1 - b_2c_2^2)$$

$$\Psi_4(u^{(1)}) = \frac{(2\pi)^2}{16} \left\{ \frac{1}{4} b_1 + c_2^2 \left[ -\frac{1}{4} b_2 - (b_1 - 1)(r + 1)(r + 3) + 3r^2 + 2b_2c_2(r + 1)^2 \right] \right\}$$

The coefficient, $4b_2$ governing the initial slope of the load-end-shortening behavior in (4.10) is plotted in Fig. 3 for rods with solid rectangular cross-section with aspect ratio, $h/t$, c.f., Fig. 1. For this cross-section, $b_1 = B_1/B_3 = (h/t)^2$, and an accurate approximation for $b_2$ from Sokolnikoff (1956) is

$$b_2 = \frac{B_2}{B_3} = \frac{2}{1 + \nu} \left\{ 1 - \frac{192}{\pi^2} \frac{t}{h} \tanh \left( \frac{\pi h}{2t} \right) \right\} \quad (4.12)$$

(For circular cross-sections, solid or concentric tubular, $b_1 = 1$ and $b_2 = 1/(1 + \nu)$.) It can be seen from the plots of $4b$ in Fig. 3 that rods having rectangular cross-sections with $h/t < 1$ ($b_1 < 1$) buckle out-of-plane with twist under increasing load in the presence of natural curvature. In other words, as in the case of the classical elastica, these rods buckle stably under prescribed load. Rods with $h/t > 1$ ($b_1 > 1$) in almost all cases buckle under decreasing load within the range of natural curvatures, $\kappa_n^2 < \kappa_n^{max}^2$, buckling out of the plane with twist. Such buckling would be unstable resulting in dynamic snapping if the axial load is prescribed. However, if the end-shortening is prescribed, buckling is stable with falling load as the end-shortening is increased beyond the onset of buckling.

The initial post-buckling analysis is valid in the limit of the classical Euler column with no natural curvature ($\kappa_n = 0$). At this limit
the above formulas give \( b = 1/8 \) such that, by (4.10),

\[
\frac{P}{P_C} = 1 + \frac{1}{2} \left( \frac{\Delta}{L} \right) + ...
\]

(4.13)

in agreement with the first term in the expansion of Euler’s closed form elastica solution. For all cases discussed in this Section, the buckling mode amplitude \( \xi \) is related to the normalized axial load and dimensionless end-shortening in the initial post-buckling regime by

\[
\xi = \pm \sqrt{\left( \frac{P}{P_C} - 1 \right) \frac{1}{b}} \quad \text{and} \quad \xi = \pm 2\sqrt{\frac{\Delta}{L}}
\]

(4.14a)

Using the expressions for the buckling mode displacements and twist in (4.3) and (4.5), one can readily generate the initial post-buckling behavior of the displacements, twist, bending moments and torque in terms of \( P/P_C \) or \( \Delta/L \).

### 4.1. Initial post-buckling behavior at the limit \( P_C = 0, \kappa_n = \kappa_{n}^{\max} \)

As has been noted, if there is no axial load, \( P = 0 \), the maximum natural curvature that a rod clamped at its two ends can sustain and still remain straight is \( \kappa_n^{\max} = 2\pi \sqrt{b_1 b_2} \). This bifurcation result was derived in Leanza et al. (2023) without establishing the stability of the bifurcation state. It is straightforward to make use of the results above to establish the stability at this limit. With this objective in mind, set \( P = 0 \) and regard the dimensionless natural curvature, \( \kappa_n \), as the new load-like parameter denoting its bifurcation value by \( \kappa_n^{\max} = 2\pi \sqrt{b_1 b_2} \). Audoly and Seffen (2015) made use of natural curvature as the load-like variable to explore the stability of solutions in their study of circular rings. With the contributions from \( \overline{P} \) in \( \Psi \) eliminated, and with \( -\Psi_2^b \) now denoting the part of \( \Psi_2 \) multiplying \( \kappa_n \) in (4.1), the Koiter analysis for the initial post-buckling coefficient \( b \) in the expansion

\[
\frac{\kappa_n}{\kappa_n^{\max}} = 1 + b\xi^2 + ... = 1 + 4b\left( \frac{\Delta}{L} \right) + ...
\]

(4.14b)

becomes

\[
b = \frac{2\Psi_n(u^{(1)})}{\kappa_n^{\max} \Psi_n^{(1)}(u^{(1)})} = \frac{1}{4b_2} \left\{ - (b_1 - 1)(r + 1)(r + 3) + b_1(r + 1)^2 + 2b_2r + 3r^2 \right\} 
\]

(4.15)

The mode used in evaluating (4.15), \( u^{(1)} = \xi(\beta^{(1)}, \omega^{(1)}, \varphi^{(1)}) \), is the same as that in the previous sub-section but with \( \overline{P} = 0 \) and \( \kappa_n = \kappa_n^{\max} = 2\pi \sqrt{b_1 b_2} \).

The coefficient \( 4b \) governing the relation between \( \kappa_n/\kappa_n^{\max} \) versus end-shortening, \( \Delta/L \), is plotted as a function of the aspect ratio, \( h/t \), of the rectangular cross-section in Fig. 4. For \( h/t > 1.15, 4b \) is negative, implying that a quasi-static bifurcation solution emanating from \( \kappa_n = \kappa_n^{\max} \) under no axial load is only possible if the natural curvature decreases. If the natural curvature is fixed, the straight state

![Fig. 4](attachment:fig4.png)

**Fig. 4.** The coefficient \( 4b \) governing the initial post-buckling relation between \( \kappa_n/\kappa_n^{\max} \) vs. \( \Delta/L \) with no axial load, \( \overline{P} = 0 \), in (4.14b) for solid rods with rectangular cross-section with aspect ratio \( h/t \) and \( \nu = 1/3 \). For a square cross-section, \( 4b = 0.519 \); for circular cross-sections with \( \nu = 1/3, 4b = 0.313 \). Also plotted is the limit value \( L\kappa_n^{C} = 2\pi \sqrt{b_1 b_2} \) for the rectangular cross-section with \( \nu = 1/3 \).
with $\kappa_n = \kappa_n^\max$ is unstable. The rod would snap dynamically to another state. Note that, with $\overline{P} = 0$, the end-shortening $\Delta$ cannot be prescribed—it is a derived quantity. Conversely, if $h/t < 1.15$, the bifurcation state is stable if the natural curvature is fixed. Further discussion of the buckling behavior for solutions originating at this limit is given in conjunction with the large amplitude solution produced in Section 5.

5. Large amplitude post-bifurcation behavior for out-of-plane displacement and twist

In this section, we compute the displacements and twist for significant departures from the straight state. We have made use of two methods to carry out these nonlinear calculations: (i) Finite Fourier expansions of $(u_1, u_3, \omega)$ together with numerical evaluation of $\overline{P}$ for any set of numerical Fourier amplitudes, and then rendering $\overline{P}$ stationary with respect to the amplitudes. and (ii) Directly solving numerically the well-known system of first order nonlinear ordinary differential equations, known as Kirchhoff rod theory, that are associated with stationarity of $\overline{P}$. Each method has advantages, but (ii) has the advantage that it generates exact solutions, subject to the accuracy of the numerical ODE method, while the accuracy of (i) depends in a complicated way on the number of Fourier amplitudes and the magnitude of the displacements and rotation. While we have generated results using both methods, the results presented in this section have all been obtained by solving the Kirchhoff rod ODEs, as has been done previously by numerous authors on other rod problems (e.g., Champneys & Thompson (1996), Audoly & Seffen (2015)). Selected details specific to the present problems are presented in the Appendix. We have employed the ODE solver BFPFD (IMSL, 2021). For problems with the possibility of multiple solutions, such as the buckling problems considered in this paper, we have made effective use of the analytical initial post-bifurcation solution as the starting estimate of the solution in the ODE solver, thereby launching the solution on the desired branch.

One specific definition for the finite amplitude solutions requires clarification. The rotation-like variable, $\tilde{\omega}$, is directly related to the twist (rotation/length) of the rod, $\kappa_2$, by $\tilde{\omega}/ds = \kappa_2$. For rods in this paper with linear torsional behavior, the torque carried by the rod is $B_2\tilde{\omega}/ds e_2$. From (3.1), it follows that $\tilde{\omega}$ differs from $\omega$, as employed until this point, by terms of order $\xi^3$. For all the prior discussion in this paper, the distinction between $\tilde{\omega}$ and $\omega$ is immaterial. However, in the Kirchhoff formulation of the ODEs for finite amplitude solutions for the clamped rod, $\tilde{\omega} = \int_0^L \kappa_2 ds$ is the variable employed and referred to as the rotation.

5.1. Examples of post-bifurcation behavior in the presence of natural curvature and axial load

The first example in Fig. 5 is for a rod with $h/t = 1/\sqrt{2}$ such that $b_1 = 1/2$. Reference to Section 3 and Fig. 2 reveals that because $b_1 < 1$ the critical buckling mode always involves out-of-plane deflection and twist, assuming $0 < \kappa_n < \kappa_n^\max$. Moreover, the bifurcation is stable with $b > 0$. Specifically, the example in Fig. 5 (with $\nu = 1/3$) has $\kappa_n^\max = 2.912$ with a natural curvature chosen as $\kappa_n = \kappa_n^\max / 2 = 1.456$ such that $\overline{P}_C = 14.80$. The initial post-bifurcation stability coefficient is $b = 0.497$. The bifurcation behavior is symmetric with respect to the center of the rod and this symmetry persists in the finite amplitude solution. The buckling behavior remains stable to loads as large as $P/P_C = 1.3$ corresponding to end-shortenings as large as $\Delta/L = 0.25$. At these levels, $\tilde{u}_3(1/2)/L \approx 0.3$ and $\tilde{\omega}(1/2) \approx -0.2\pi$. (Note that the displacement components, $\tilde{u}_1$ and $\tilde{u}_3$, in Figs. 5–8 are defined with respect to the fixed Cartesian axes and not relative to the embedded base vectors.) The 3D plot of the shape of the deforming rod in Fig. 5c shows that it curls as it buckles. The symmetry of the pre-buckled rod with respect to the $(i_1, i_2)$ plane means that the solution plotted in Fig. 5 would pertain equally if the signs of both $\tilde{u}_3(x)$ and $\tilde{\omega}(x)$ were switched with the sign of $\tilde{u}_1(x)$ unchanged. The agreement between the finite amplitude results and the initial post-bifurcation predictions depends on the variable in question, but it is clearly excellent for sufficiently small.

Fig. 5. Post-bifurcation behavior for a rod with a solid rectangular cross-section ($h/t = 1/\sqrt{2}$, $b_1 = B_1/B_3 = 1/2$, $\nu = 1/3$) with natural curvature $\kappa_n = 1.456$. The bifurcation for this case is stable ($b = 0.497$) with a mode which combines out-of-plane displacement with twist. a) Load vs. end-shortening. b) Load vs. two displacement components and the rotation at the center of the rod. c) A 3D plot of the shape of the rod at an end-shortening of $\Delta/L = 0.2$. Included in parts a) and b) are the predictions of the initial post-buckling expansion ($b = 0.497$), shown as dashed lines.
departures from the straight state. The in-plane displacement $u_1$ is of order $\xi^2$ and its initial post-bifurcation coefficient has not been evaluated.

The second example illustrates the post-buckling behavior for a case where the load decreases as the buckle grows. In this case, $h/t = \sqrt{2}$, $b_1 = 2$, $\nu = 1/3$. The natural curvature $\kappa_n = 6.993$. The bifurcation mode for this case is again a mode which combines out-of-plane displacement with twist but in this case the bifurcation is unstable if the load is prescribed ($b = -2.354$). a) Load vs. end-shortening. b) Load vs. two displacement components and rotation at the center of the rod. c) 3D plot of the rod shape at $\Delta/L = 0.147$. The initial post-bifurcation predictions are shown as dashed lines.

Fig. 6. Post-bifurcation behavior for a rod with a solid rectangular cross-section ($h/t = \sqrt{2}$, $b_1 = 2$, $\nu = 1/3$) with natural curvature $\kappa_n = 6.993$. The bifurcation mode for this case is again a mode which combines out-of-plane displacement with twist but in this case the bifurcation is unstable if the load is prescribed ($b = -2.354$). a) Load vs. end-shortening. b) Load vs. two displacement components and rotation at the center of the rod. c) 3D plot of the rod shape at $\Delta/L = 0.147$. The initial post-bifurcation predictions are shown as dashed lines.

Fig. 7. A rod with square cross-section ($h/t = 1$, $\nu = 1/3$) and the dimensionless natural curvature $\kappa_n = \kappa_n^{\text{max}} = 5.014$ is straightened and clamped as in Fig. 1 with no axial load $P$ allowed to develop. Solutions are obtained by treating the natural curvature $\kappa_n$ as a load-like variable along the post-bifurcation solution path. a) Natural curvature vs. end-shortening. b) Two displacement components and the rotation at the center of the rod. c) A 3D rendering of the shape of the rod when $\kappa_n/\kappa_n^C = 1.4$ corresponding to $\Delta/L = 0.729$. The initial post-bifurcation predictions ($b = 0.1296$) are shown as dashed lines.

Fig. 7. A rod with square cross-section ($h/t = 1$, $\nu = 1/3$) and the dimensionless natural curvature $\kappa_n = \kappa_n^{\text{max}} = 5.014$ is straightened and clamped as in Fig. 1 with no axial load $P$ allowed to develop. Solutions are obtained by treating the natural curvature $\kappa_n$ as a load-like variable along the post-bifurcation solution path. a) Natural curvature vs. end-shortening. b) Two displacement components and the rotation at the center of the rod. c) A 3D rendering of the shape of the rod when $\kappa_n/\kappa_n^C = 1.4$ corresponding to $\Delta/L = 0.729$. The initial post-bifurcation predictions ($b = 0.1296$) are shown as dashed lines.

The second example illustrates the post-buckling behavior for a case where the load decreases as the buckle grows. In this case, $h/t = \sqrt{2}$ such that $b_1 = 2$ and (with $\nu = 1/3$), $\kappa_n = 5.792$ and $\kappa_n^{\text{max}} = 8.192$. The natural curvature of the rod in Fig. 6 has been chosen as $\kappa_n = 6.993$, lying halfway between $\kappa_n^C$ and $\kappa_n^{\text{max}}$, such that $P_C = 21.43$ and $b = -2.354$. The solution plotted in Fig. 6 again has symmetry about the center of the rod and symmetry with respect to sign interchanges of $u_3$ and $\omega$. The solution has been computed until $P = 0$ at which $\Delta/L = 0.147$. The approximation based on the initial-post-bifurcation expansion (the dashed lines in Fig. 6) remains reasonably accurate for relatively large deflections for this case. If the load, $P$, is prescribed, then the straight bifurcation state at $P = P_C$ is unstable because any perturbation of the rod in the form of the bifurcation mode could not be supported. Under prescribed $P$, the entire solution plotted in Fig. 6 is unstable equilibrium. However, if the end-shortening, $\Delta$, is prescribed, the load-drop is determined by $\Delta$, and the solution is stable, assuming bifurcation from the symmetric solution does not occur (which we have not carefully investigated).

5.2. Examples of post-bifurcation behavior driven by natural curvature with no axial load

In this sub-section, we consider the limit when the dimensionless natural curvature attains $\kappa_n^{\text{max}}$ and no axial load is permitted to develop. Now, $\kappa_n$ is taken as the ‘load parameter’ in the bifurcation process. The coefficient $b$ determining the initial post-bifurcation
behavior of solutions as buckling progresses was derived and presented in Section 4.1. A large amplitude solution is presented in Fig. 7 starting with the case of a square cross-section, $h/t = 1$ with $\nu = 1/3$, for which $\kappa_{n\text{max}} = 5.014$ and $b = 0.1296$. As noted earlier, this case is stable in the straight state. If the natural curvature is increased above $\kappa_{n\text{max}}$, buckling takes place stably with a continuous increase of the buckling displacements and rotation, as seen in Fig. 7. In this example, the natural curvature is increased to the level $\kappa_{n} / \kappa_{n\text{max}} = 1.4$, where the end-shortening reaches $\Delta/L = 0.729$ and the rod undergoes quite large displacements and rotations. These calculations have been carried out by imposing symmetry about the middle of the rod, and, thus, we have not explored whether asymmetric bifurcations occur at some point of the history shown.

The stable behavior exhibited by the rod with a square cross-section is in contrast to the highly unstable behavior expected from a rod with a rectangular cross-section ($h/t = \sqrt{2}$, $\nu = 1/3$) under no axial load and with a dimensionless natural curvature given by $\kappa_{n\text{max}} = 8.192$. The post-bifurcation coefficient is $b = -0.3472$. As seen in Fig. 8, the solution bifurcating at $\kappa_{n} = \kappa_{n\text{max}}$ has decreasing $\kappa_{n}$ along the solution path. Thus, if the rod has natural curvature fixed at $\kappa_{n} = \kappa_{n\text{max}}$, it will be unstable because any perturbation directed along the solution path could not be maintained by the rod, and it would necessarily snap dynamically to some other state. It is important to appreciate that requiring $P = 0$ eliminates the possibility of prescribing values of end-shortening because $\Delta$ becomes a derived quantity and part of the solution. Under prescribed natural curvature, the solution in Fig. 8 is unstable over the entire range plotted. At $\kappa_{n} / \kappa_{n\text{max}} = 0.7$ (and $\Delta/L = 0.604$), the shape of the rod is similar to that observed for the previous example.

### 6. Experimental examples of the modes of buckling

In this section, we briefly explore experiments on three rods of different natural curvatures and geometries to highlight their distinct buckling behaviors. The first example, as shown in Fig. 9a, is of a rod that is straight in the natural state ($\kappa_{n} = 0$) with $h/t = 1.45$. As expected, because the bending stiffness for out-of-plane bending exceeds that for in-plane bending ($B_1 = B_1/B_3 > 1$), the axially compressed rod buckles in-plane, i.e., in the $(i_1, i_2)$-plane. After the onset of buckling, the load steadily increases as seen in the load vs end-shortening plot in Fig. 9a. As discussed in Section 4, this is classical Euler buckling behavior, and the initial post-buckling load-end shortening behavior is given by (4.13), i.e., $P/P_C = 1 + (\Delta/L)/2 + \ldots$, which is included as a dashed line in Fig. 9a in reasonably good agreement with the experimental results. This experiment is shown in Video 1 of the Supplementary Materials.

The next two cases, involving natural curvature, illustrate how the buckling behavior is heavily influenced by $h/t$ and $\kappa_{n}$. The example shown in Fig. 9b is for a rod with $\kappa_{n} = 1.75$ and $h/t = 0.75$. The natural curvature, which is depicted on the left, falls well within the range, $0 < \kappa_{n} < \kappa_{n\text{max}}$, such that the rod will remain straight under no axial load. This is a case in which the out-of-plane bending stiffness is less than in-plane stiffness ($B_1 = B_1/B_3 < 1$), so it is predicted to buckle out-of-plane, i.e., in the $(i_2, i_3)$-plane. Due to the natural curvature, the rod also twists (see Video 2 of the Supplementary Materials). As buckling ensues, the rod reveals strong post-buckling load carrying capacity, analogous to that in Fig. 5, with the load increasing to a value of $P/P_C \approx 1.70$ at an end-shortening of $\Delta/L = 0.25$. The value of the initial post-buckling coefficient for this case is $b = 0.516$ such that, by (4.10), $P/P_C = 1 + 2.06 (\Delta/L) + \ldots$, in fair agreement with the experimental results in Fig. 9b. The last example in Fig. 9c illustrates a case, analogous to that in Fig. 6, where the load falls sharply after the onset of buckling. The rod shown, with $h/t = 1.42$ and $\kappa_{n} = 6.03$ ($\kappa_{n} < \kappa_{n\text{max}}$), buckles out-of-the plane of its natural curvature with twist, as predicted (see Video 3 of the Supplementary Materials). The load decreases sharply as buckling progresses, to a value of $P/P_C \approx 0.3$ at an end-shortening of $\Delta/L = 0.25$. For this case, the initial post-bifurcation analysis of Section 4 gives $b = -1.211$ and $P/P_C = 1 - 4.84 (\Delta/L) + \ldots$, which, as seen in Fig. 9c, is again in fair agreement with the experimental results in its essential details at least at smaller buckling amplitudes. All three cases in Fig. 9 have small initial geometric imperfections. That is, they do not have perfectly uniform curvature in the natural state and they may have slight non-
uniformities in their cross-section such that they are not perfectly straight when inserted into the test apparatus. These factors almost certainly explain some of the discrepancy between the initial post-bifurcation prediction and the experimental results.

As discussed earlier, the behavior in Fig. 9c would be unstable if the load $P$ had been prescribed. In these experiments, the test machine is exceedingly stiff compared to the specimen such that, effectively, the end-shortening is prescribed. The rich buckling behaviors outlined in previous sections are well-captured experimentally. Finite element analysis (FEA) was additionally performed.

Fig. 9. Experimental results for the buckling of three rods, each with $L = 175$ mm, $\nu = 1 / 3$, and different $\kappa_n$ and $h / t$. Schematics of the rods’ natural curvatures are shown on the left. In the central section, experimental views of the rods at the end-shortening $\Delta / L = 0.25$, are shown in the $(i_1, i_2)$ plane and the $(i_2, i_3)$ plane. On the right are the experimental load vs. end-shortening plots for rods with a) $\kappa_n = 0$, $h / t = 1.45$, b) $\kappa_n = 1.75$, $h / t = 0.75$, and c) $\kappa_n = 6.03$, $h / t = 1.42$. Scale bars: 20 mm. The solid lines are the experimental measurements, and the dashed lines are the results predicted by the initial post-bifurcation analysis.
for all three rods to calibrate the positioning of the curves in the load vs end-shortening plots and to further compare to the theoretical predictions. The FEA results, in addition to some further experimental details, are provided in Appendix B.

7. Concluding Remarks

Natural curvature in the elastica problem significantly enriches the variety of buckling behaviors exhibited by the rod or beam. While there is a subset of the parameter space for which the buckling behavior remains two-dimensional and identical to classical Euler buckling, three-dimensional behavior, including twisting, occurs in much of the space. These 3D problems have been addressed within the framework of Kirchhoff rod theory. The initial post-bifurcation approach of W.T. Koiter, which can be carried out analytically in closed form for the extended elastica problems, reveals the nature of the stability of the bifurcation as dependent on the geometry of the rod, on its natural curvature, and on whether the load or the rod end-shortening is prescribed. While the classical elastica displays stable post-buckling behavior, the elastica with natural curvature exhibits a wide range of behaviors from stable to highly unstable. The stability of the interesting limit in which the maximum uniform natural curvature is imposed on a rod or beam clamped at both ends has also been determined via the initial post-bifurcation analysis. Using the exact Kirchhoff formulation, we have also carried out numerical simulations characterizing relatively large buckling deflections and rotations. These further emphasize the three-dimensionality produced by the introduction of natural curvature. The brief experimental section provides illustrations of the main features revealed by the analysis, including an example with highly stable out-of-plane buckling with twist, and an example with a dramatic loss of load carrying capacity as buckling occurs leading to stable behavior under the prescribed end-shortening conditions of the test, but it would display highly unstable buckling had the test apparatus prescribed the load.

Leonhard Euler published more than thirty papers on the elastica with the first appearing almost 300 years ago. It was not uncommon for Euler to return multiple times to a problem he had solved to improve the presentation and to add new aspects. Kirchhoff’s classic paper on three-dimensional rod theory dates from 1859. With such notable precedents, we readily admit that this paper only scratches the surface of the elastica problem with natural curvature. It is possible that closed-form analytical solutions exist for large amplitude buckling, analogous to those Euler obtained for the two-dimensional problem. While unstable states have been detailed in this paper which lead to dynamic snapping, we have not determined the arrest states of the snapping rod. These states lie outside the range of states we have explored and presented. In addition, our study has been limited to buckling modes that are symmetric about the center of the rod, and, while all the experiments we conducted displayed this symmetry, it is likely to be lost in some cases at sufficiently large buckling amplitudes. Almost certainly, there is a complex and rich set of states awaiting discovery.

CRediT authorship contribution statement

Sophie Leanza: Writing – review & editing, Visualization, Investigation, Data curation. Ruike Renee Zhao: Writing – review & editing, Supervision, Investigation, Data curation. John W. Hutchinson: Writing – original draft, Supervision, Methodology, Investigation, Formal analysis, Data curation, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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Supplementary materials

Supplementary material associated with this article can be found, in the online version, at doi:10.1016/j.jmps.2024.105690.

Appendix

A. Selected details of the finite amplitude buckling simulations

An extensive literature exists on Kirchhoff rod theory, its applications, and its numerical implementation (e.g., Antman, 1995, Champneys and Thompson, 1996, Audoly and Seffen, 2015), which will not be duplicated here. Here we include a few brief comments that are specifically relevant to the present clamped rod study. Let \( \Omega = \kappa \mathbf{e}_i \) (sum on i) be the curvature vector such that the
Frenet-Seret formula is \( \mathbf{e}_i = \mathbf{\Omega} \times \mathbf{e}_i \) with \( ( \mathbf{\Gamma} = d( \mathbf{\gamma}) ) ds \). The force and moment carried at any point along the rod are \( \mathbf{F} = F_\mathbf{e}_i \) and \( \mathbf{M} = M_i \mathbf{e}_i \). Because the rod has no external forces or moments acting on it other than at its ends, equilibrium at every point along the rod requires \( \mathbf{F} = 0 \) and \( \mathbf{M} = r' \times \mathbf{F} = 0 \). The components of the moment and curvature are related by \( (M_1 = B_1 \kappa_1, M_2 = B_2 \kappa_2, M_3 = B_3 (\kappa_3 - \kappa_n)) \). One must also include in the set of first order ODEs describing Kirchhoff rod theory the equations for the displacement vector, \( \mathbf{u}' = e_2 - i_2 \), i.e., \( u_i e_i = e_2 - i_2 - u_3 \mathbf{\Omega} \times \mathbf{e}_i \), and that for \( i_2 = c^{(2)}_i \mathbf{e}_i \) as \( c^{(2)}_i \mathbf{e}_i = -c^{(2)}_i \mathbf{\Omega} \times \mathbf{e}_i \). Because we will want to express the components of the displacement vector in terms of the fixed Cartesian base vectors, \( (i_1, i_2, i_3) \), three additional equations for \( i_1 = c^{(1)}_i \mathbf{e}_i \) are included, i.e., \( c^{(1)}_i \mathbf{e}_i = -c^{(1)}_i \mathbf{\Omega} \times \mathbf{e}_i \), which then gives the components of \( i_3 = i_1 \times i_2 \).

For the rod under axial compression, the governing equations reduce to 15 first order ODEs in the form \( dy_i/ds = f_i(y) \) where

\[
y_i = (F_1, F_2, F_3, M_1, M_2, M_3, u_1, u_2, u_3, c^{(2)}_1, c^{(2)}_2, c^{(2)}_3, c^{(1)}_1, c^{(1)}_2, c^{(1)}_3)
\]

There are 10 boundary conditions at the left end of the rod,

\[
(M_3 = 0, u_1 = 0, u_2 = 0, u_3 = 0, c^{(2)}_1 = 0, c^{(2)}_2 = 1, c^{(2)}_3 = 0, c^{(1)}_1 = 1, c^{(1)}_2 = 0, c^{(1)}_3 = 0),
\]

while at the right end there are five,

\[
(u_1 = 0, u_2 = 0, u_3 = -\Delta, c^{(2)}_1 = 0, c^{(2)}_3 = 0).
\]

The axial load is evaluated from \( P = -F_2 \) at either end. Alternatively, the three equations for \( c^{(1)}_1 \) could be deleted from the set of 15 equations and then, after the solution to the remaining 12 equations is found, they could be solved to obtain \( c^{(1)}_1 \).

The above system of equations makes no assumptions on symmetry of the solution with respect to the center of the rod (of course, \( u_2 \) is not symmetric about the center, but \( u'_2 \) is). In all the examples presented in the paper for axial compression, the numerical solutions obtained were symmetric. If symmetry is assumed, it follows that \( F = -P_1 = -Pc^{(3)}_1 \mathbf{e}_n \) and the system can be simplified accordingly. For the problems in Section 5.2 with \( P = 0 \), we have assumed symmetry about the center of the rod. By exploiting symmetry, the system can be reduced to 13 first order ODEs with

\[
y_i = (M_1, M_2, M_3, u_1, u_2, u_3, c^{(2)}_1, c^{(2)}_2, c^{(2)}_3, c^{(1)}_1, c^{(1)}_2, c^{(1)}_3, \kappa_n)
\]

Now, \( \kappa_n \) is treated as an unknown satisfying \( d\kappa_n/ds = 0 \). The 10 boundary conditions at \( s = 0 \) are

\[
(M_3 = 0, u_1 = 0, u_2 = 0, u_3 = 0, c^{(2)}_1 = 0, c^{(2)}_2 = 1, c^{(2)}_3 = 0, c^{(1)}_1 = 1, c^{(1)}_2 = 0, c^{(1)}_3 = 0),
\]

and the 3 conditions at \( s = L/2 \) are

\[
(u_3 = -\Delta/2, c^{(2)}_1 = 0, c^{(2)}_3 = 0).
\]

### B. Selected details of the experimental procedures

**Experiment:** All rods were printed using an Ultimaker S5 (Ultimaker, Netherlands) with Tough PLA, whose Young’s Modulus was measured as \( E = 2.21 \text{ GPa} \) via a tensile test. The Poisson’s ratio of the PLA was assumed to be \( \nu = 1/3 \). All axial compression tests were carried out using a universal testing machine (3344, Instron, Inc., USA), in which the rods were compressed at a rate of 0.01 \( \text{L/s} \). The samples were straightened and clamped into the testing machine, where they were held for only 15 s before being loaded, to mitigate viscoelastic effects. The specific dimensions of the rods discussed in Fig. 9 are summarized in Table B1 below. FEA: The commercial software ABAQUS 2021 (Dassault Systèmes, France) was used to predict the buckling of the rods studied in Section 6 under axial shortening. For all simulations, the C3D8 linear brick element was used, with six and eight elements assigned through the shorter and longer dimensions of the cross-section, respectively, and 800 elements along the length of the rod. An isotropic elastic model was adopted for the rods, with the Young’s modulus as 2.21 GPa and a Poisson’s ratio of 1/3. Boundary conditions were applied to reference points constrained with nodes of the rod’s cross-section using multi-point constraints (MPCs). A small perturbation (0.01 – 0.05 mm displacement) was introduced prior to prescribed end-shortening to trigger buckling. The FEA results are summarized below in Fig. B1.

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<td>Experimental rod dimensions.</td>
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<td>Fig. 9c</td>
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Fig. B1. FEA (top row) and experimental (bottom row) results of the buckling of the rods shown in Fig. 9 upon $\Delta / L = 0.25$ end-shortening. Views of the $(i_1, i_2)$ plane and the $(i_2, i_3)$ plane for rods with a) $\kappa_n = 0$, $h / t = 1.45$, b) $\kappa_n = 1.75$, $h / t = 0.75$, and c) $\kappa_n = 6.03$, $h / t = 1.42$. Scale bars: 10 mm.

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